5.5 Multiple Angle and Product-to-Sum Formulas

What you should learn
• Use multiple-angle formulas to rewrite and evaluate trigonometric functions.
• Use power-reducing formulas to rewrite and evaluate trigonometric functions.
• Use half-angle formulas to rewrite and evaluate trigonometric functions.
• Use product-to-sum and sum-to-product formulas to rewrite and evaluate trigonometric functions.
• Use trigonometric formulas to rewrite real-life models.

Why you should learn it
You can use a variety of trigonometric formulas to rewrite trigonometric functions in more convenient forms. For instance, in Exercise 119 on page 417, you can use a double-angle formula to determine at what angle an athlete must throw a javelin.

Multiple-Angle Formulas
In this section, you will study four other categories of trigonometric identities.

1. The first category involves functions of multiple angles such as \( \sin k u \) and \( \cos k u \).
2. The second category involves squares of trigonometric functions such as \( \sin^2 u \).
3. The third category involves functions of half-angles such as \( \sin(u/2) \).
4. The fourth category involves products of trigonometric functions such as \( \sin u \cos v \).

You should learn the **double-angle formulas** because they are used often in trigonometry and calculus. For proofs of the formulas, see Proofs in Mathematics on page 425.

### Double-Angle Formulas

\[
\begin{align*}
\sin 2u &= 2 \sin u \cos u \\
\cos 2u &= \cos^2 u - \sin^2 u \\
\tan 2u &= \frac{2 \tan u}{1 - \tan^2 u} \\
\cos 2u &= 1 - 2 \sin^2 u
\end{align*}
\]

**Example 1** Solving a Multiple-Angle Equation

Solve \( 2 \cos x + \sin 2x = 0 \).

**Solution** Begin by rewriting the equation so that it involves functions of \( x \) (rather than \( 2x \)). Then factor and solve as usual.

\[
2 \cos x + \sin 2x = 0
\]

Write original equation.

\[
2 \cos x + 2 \sin x \cos x = 0
\]

Double-angle formula

\[
2 \cos x(1 + \sin x) = 0
\]

Factor.

\[
2 \cos x = 0 \quad \text{and} \quad 1 + \sin x = 0
\]

Set factors equal to zero.

\[
x = \frac{\pi}{2}, \quad x = \frac{3\pi}{2}
\]

Solutions in \([0, 2\pi)\)

So, the general solution is

\[
x = \frac{\pi}{2} + 2n \pi \quad \text{and} \quad x = \frac{3\pi}{2} + 2n \pi
\]

where \( n \) is an integer. Try verifying these solutions graphically.

Now try Exercise 9.
Example 2  Using Double-Angle Formulas to Analyze Graphs

Use a double-angle formula to rewrite the equation
\[ y = 4 \cos^2 x - 2. \]

Then sketch the graph of the equation over the interval \([0, 2\pi]\).

Solution

Using the double-angle formula for \(\cos 2u\), you can rewrite the original equation as
\[ y = 4 \cos^2 x - 2 \quad \text{Write original equation.} \]
\[ = 2(2 \cos^2 x - 1) \quad \text{Factor.} \]
\[ = 2 \cos 2x. \quad \text{Use double-angle formula.} \]

Using the techniques discussed in Section 4.5, you can recognize that the graph of this function has an amplitude of 2 and a period of \(\pi\). The key points in the interval \([0, \pi]\) are as follows.

<table>
<thead>
<tr>
<th>Maximum</th>
<th>Intercept</th>
<th>Minimum</th>
<th>Intercept</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 2))</td>
<td>((\pi/4, 0))</td>
<td>((\pi/2, -2))</td>
<td>((3\pi/4, 0))</td>
<td>((\pi, 2))</td>
</tr>
</tbody>
</table>

Two cycles of the graph are shown in Figure 5.9.

Example 3  Evaluating Functions Involving Double Angles

Use the following to find \(\sin 2\theta\), \(\cos 2\theta\), and \(\tan 2\theta\).
\[ \cos \theta = \frac{5}{13}, \quad \frac{3\pi}{2} < \theta < 2\pi \]

Solution

From Figure 5.10, you can see that \(\sin \theta = y/r = -12/13\). Consequently, using each of the double-angle formulas, you can write
\[ \sin 2\theta = 2 \sin \theta \cos \theta = 2 \left( -\frac{12}{13} \right) \left( \frac{5}{13} \right) = -\frac{120}{169} \]
\[ \cos 2\theta = 2 \cos^2 \theta - 1 = 2 \left( \frac{25}{169} \right) - 1 = -\frac{119}{169} \]
\[ \tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{120}{119} \]

The double-angle formulas are not restricted to angles \(2\theta\) and \(\theta\). Other double combinations, such as \(4\theta\) and \(2\theta\) or \(6\theta\) and \(3\theta\), are also valid. Here are two examples.
\[ \sin 4\theta = 2 \sin 2\theta \cos 2\theta \quad \text{and} \quad \cos 6\theta = \cos^3 3\theta - \sin^2 3\theta \]

By using double-angle formulas together with the sum formulas given in the preceding section, you can form other multiple-angle formulas.
**Example 4** Deriving a Triple-Angle Formula

\[
\sin 3x = \sin(2x + x)
\]

\[
= \sin 2x \cos x + \cos 2x \sin x
\]

\[
= 2 \sin x \cos x \cos x + (1 - 2 \sin^2 x) \sin x
\]

\[
= 2 \sin x \cos^2 x + \sin x - 2 \sin^3 x
\]

\[
= 2 \sin x(1 - \sin^2 x) + \sin x - 2 \sin^3 x
\]

\[
= 2 \sin x - 2 \sin^3 x + \sin x - 2 \sin^3 x
\]

\[
= 3 \sin x - 4 \sin^3 x
\]

**CHECKPOINT** Now try Exercise 97.

### Power-Reducing Formulas

The double-angle formulas can be used to obtain the following power-reducing formulas. Example 5 shows a typical power reduction that is used in calculus.

**Power-Reducing Formulas**

\[
\sin^2 u = \frac{1 - \cos 2u}{2} \quad \cos^2 u = \frac{1 + \cos 2u}{2} \quad \tan^2 u = \frac{1 - \cos 2u}{1 + \cos 2u}
\]

For a proof of the power-reducing formulas, see Proofs in Mathematics on page 425.

**Example 5** Reducing a Power

Rewrite \( \sin^4 x \) as a sum of first powers of the cosines of multiple angles.

**Solution**

Note the repeated use of power-reducing formulas.

\[
\sin^4 x = (\sin^2 x)^2
\]

\[
= \left( \frac{1 - \cos 2x}{2} \right)^2 \quad \text{Property of exponents}
\]

\[
= \frac{1}{4} (1 - 2 \cos 2x + \cos^2 2x) \quad \text{Power-reducing formula}
\]

\[
= \frac{1}{4} \left( 1 - 2 \cos 2x + \frac{1 + \cos 4x}{2} \right) \quad \text{Expand.}
\]

\[
= \frac{1}{4} \left( 1 - 2 \cos 2x + \frac{1}{8} + \frac{1}{8} \cos 4x \right) \quad \text{Power-reducing formula}
\]

\[
= \frac{1}{4} \left( 3 - 4 \cos 2x + \cos 4x \right) \quad \text{Distributive Property}
\]

\[
= \frac{1}{8} (3 - 4 \cos 2x + \cos 4x) \quad \text{Factor out common factor.}
\]

**CHECKPOINT** Now try Exercise 29.
Half-Angle Formulas

You can derive some useful alternative forms of the power-reducing formulas by replacing \( u \) with \( \frac{u}{2} \). The results are called half-angle formulas.

**Half-Angle Formulas**

\[
\sin \frac{u}{2} = \pm \sqrt{\frac{1 - \cos u}{2}} \\
\cos \frac{u}{2} = \pm \sqrt{\frac{1 + \cos u}{2}} \\
\tan \frac{u}{2} = \frac{1 - \cos u}{\sin u} = \frac{\sin u}{1 + \cos u}
\]

The signs of \( \sin \frac{u}{2} \) and \( \cos \frac{u}{2} \) depend on the quadrant in which \( \frac{u}{2} \) lies.

---

**Example 6** Using a Half-Angle Formula

Find the exact value of \( \sin 105^\circ \).

**Solution**

Begin by noting that \( 105^\circ \) is half of \( 210^\circ \). Then, using the half-angle formula for \( \sin(u/2) \) and the fact that \( 105^\circ \) lies in Quadrant II, you have

\[
\sin 105^\circ = \sqrt{\frac{1 - \cos 210^\circ}{2}} \\
= \sqrt{\frac{1 - (-\cos 30^\circ)}{2}} \\
= \sqrt{\frac{1 + (\sqrt{3}/2)}{2}} \\
= \frac{\sqrt{2 + \sqrt{3}}}{2}.
\]

The positive square root is chosen because \( \sin \theta \) is positive in Quadrant II.

*CHECKPOINT* Now try Exercise 41.

Use your calculator to verify the result obtained in Example 6. That is, evaluate \( \sin 105^\circ \) and \( (\sqrt{2 + \sqrt{3}})/2 \).

\[
\sin 105^\circ = 0.9659258 \\
\frac{\sqrt{2 + \sqrt{3}}}{2} = 0.9659258
\]

You can see that both values are approximately 0.9659258.
Example 7  Solving a Trigonometric Equation

Find all solutions of $2 - \sin^2 x = 2 \cos^2 x$ in the interval $[0, 2\pi]$.

**Algebraic Solution**

\[
2 - \sin^2 x = 2 \cos^2 x
\]
Write original equation.

\[
2 - \sin^2 x = 2 \left( \pm \sqrt{\frac{1 + \cos x}{2}} \right)^2
\]
Half-angle formula

\[
2 - \sin^2 x = 2 \left( \frac{1 + \cos x}{2} \right)
\]
Simplify.

\[
2 - \sin^2 x = 1 + \cos x
\]
Simplify.

\[
2 - (1 - \cos^2 x) = 1 + \cos x
\]
Pythagorean identity

\[
\cos^2 x - \cos x = 0
\]
Simplify.

\[
\cos x(\cos x - 1) = 0
\]
Factor.

By setting the factors $\cos x$ and $\cos x - 1$ equal to zero, you find that the solutions in the interval $[0, 2\pi]$ are

\[
x = \frac{\pi}{2}, \quad x = \frac{3\pi}{2}, \quad \text{and} \quad x = 0.
\]

*CHECKPOINT*  Now try Exercise 59.

**Graphical Solution**

Use a graphing utility set in **radian** mode to graph $y = 2 - \sin^2 x - 2 \cos^2 x$, as shown in Figure 5.11. Use the **zero** or **root** feature or the **zoom** and **trace** features to approximate the x-intercepts in the interval $[0, 2\pi]$ to be

\[
x = 0, \quad x \approx 1.571 = \frac{\pi}{2}, \quad \text{and} \quad x \approx 4.712 = \frac{3\pi}{2}.
\]

These values are the approximate solutions of $2 - \sin^2 x - 2 \cos^2 x = 0$ in the interval $[0, 2\pi]$.

![FIGURE 5.11](NULL)

Product-to-Sum Formulas

Each of the following **product-to-sum formulas** is easily verified using the sum and difference formulas discussed in the preceding section.

**Product-to-Sum Formulas**

\[
\sin u \sin v = \frac{1}{2} [\cos (u - v) - \cos (u + v)]
\]

\[
\cos u \cos v = \frac{1}{2} [\cos (u - v) + \cos (u + v)]
\]

\[
\sin u \cos v = \frac{1}{2} [\sin (u + v) + \sin (u - v)]
\]

\[
\cos u \sin v = \frac{1}{2} [\sin (u + v) - \sin (u - v)]
\]

Product-to-sum formulas are used in calculus to evaluate integrals involving the products of sines and cosines of two different angles.
Writing Products as Sums

Rewrite the product as a sum or difference.

Solution
Using the appropriate product-to-sum formula, you obtain
\[ \cos 5x \sin 4x = \frac{1}{2} [\sin(5x + 4x) - \sin(5x - 4x)] = \frac{1}{2} \sin 9x - \frac{1}{2} \sin x. \]

Now try Exercise 67.

Occasionally, it is useful to reverse the procedure and write a sum of trigonometric functions as a product. This can be accomplished with the following sum-to-product formulas.

\[
\begin{align*}
\sin u + \sin v &= 2 \sin \left(\frac{u + v}{2}\right) \cos \left(\frac{u - v}{2}\right) \\
\sin u - \sin v &= 2 \cos \left(\frac{u + v}{2}\right) \sin \left(\frac{u - v}{2}\right) \\
\cos u + \cos v &= 2 \cos \left(\frac{u + v}{2}\right) \cos \left(\frac{u - v}{2}\right) \\
\cos u - \cos v &= -2 \sin \left(\frac{u + v}{2}\right) \sin \left(\frac{u - v}{2}\right)
\end{align*}
\]

For a proof of the sum-to-product formulas, see Proofs in Mathematics on page 426.

Example 8 Writing Products as Sums

Rewrite the product \( \cos 5x \sin 4x \) as a sum or difference.

Solution
Using the appropriate product-to-sum formula, you obtain
\[ \cos 5x \sin 4x = \frac{1}{2} [\sin(5x + 4x) - \sin(5x - 4x)] = \frac{1}{2} \sin 9x - \frac{1}{2} \sin x. \]

Now try Exercise 67.

Example 9 Using a Sum-to-Product Formula

Find the exact value of \( \cos 195^\circ + \cos 105^\circ \).

Solution
Using the appropriate sum-to-product formula, you obtain
\[ \cos 195^\circ + \cos 105^\circ = 2 \cos \left(\frac{195^\circ + 105^\circ}{2}\right) \cos \left(\frac{195^\circ - 105^\circ}{2}\right) = 2 \cos 150^\circ \cos 45^\circ \]
\[ = 2 \left(-\frac{\sqrt{3}}{2}\right) \left(\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{6}}{2}. \]

Now try Exercise 83.
Solving a Trigonometric Equation

Solve the equation $\sin 5x + \sin 3x = 0$.

Solution

$$\sin 5x + \sin 3x = 0$$  Write original equation.

$$2 \sin \left(\frac{5x + 3x}{2}\right) \cos \left(\frac{5x - 3x}{2}\right) = 0$$  Sum-to-product formula

$$2 \sin 4x \cos x = 0$$  Simplify.

By setting the factor $2 \sin 4x$ equal to zero, you can find that the solutions in the interval $[0, 2\pi]$ are

$$x = 0, \pi, \frac{3\pi}{4}, \frac{\pi}{2}, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}.$$  

The equation $\cos x = 0$ yields no additional solutions, and you can conclude that the solutions are of the form $x = \frac{n\pi}{4}$, where $n$ is an integer. You can confirm this graphically by sketching the graph of $y = \sin 5x + \sin 3x$, as shown in Figure 5.12. From the graph you can see that the $x$-intercepts occur at multiples of $\pi/4$.

Checkpoint  Now try Exercise 87.

Example 11  Verifying a Trigonometric Identity

Verify the identity

$$\frac{\sin t + \sin 3t}{\cos t + \cos 3t} = \tan 2t.$$  

Solution

Using appropriate sum-to-product formulas, you have

$$\frac{\sin t + \sin 3t}{\cos t + \cos 3t} = \frac{2 \sin \left(\frac{t + 3t}{2}\right) \cos \left(\frac{t - 3t}{2}\right)}{2 \cos \left(\frac{t + 3t}{2}\right) \cos \left(\frac{t - 3t}{2}\right)}$$

$$= \frac{2 \sin(2t) \cos(-t)}{2 \cos(2t) \cos(-t)}$$

$$= \frac{\sin 2t}{\cos 2t}$$

$$= \tan 2t.$$  

Checkpoint  Now try Exercise 105.
Application

**Example 12  Projectile Motion**

Ignoring air resistance, the range of a projectile fired at an angle \( \theta \) with the horizontal and with an initial velocity of \( v_0 \) feet per second is given by

\[
r = \frac{1}{16} v_0^2 \sin \theta \cos \theta
\]

where \( r \) is the horizontal distance (in feet) that the projectile will travel. A place kicker for a football team can kick a football from ground level with an initial velocity of 80 feet per second (see Figure 5.13).

a. Write the projectile motion model in a simpler form.

b. At what angle must the player kick the football so that the football travels 200 feet?

c. For what angle is the horizontal distance the football travels a maximum?

**Solution**

a. You can use a double-angle formula to rewrite the projectile motion model as

\[
r = \frac{1}{32} v_0^2 (2 \sin \theta \cos \theta)
\]

Rewrite original projectile motion model.

\[
= \frac{1}{32} v_0^2 \sin 2\theta.
\]

Rewrite model using a double-angle formula.

b. \( r = \frac{1}{32} v_0^2 \sin 2\theta \)

Write projectile motion model.

\[
200 = \frac{1}{32} (80)^2 \sin 2\theta
\]

Substitute 200 for \( r \) and 80 for \( v_0 \).

\[
200 = 200 \sin 2\theta
\]

Simplify.

\[
1 = \sin 2\theta
\]

Divide each side by 200.

You know that \( 2\theta = \pi/2 \), so dividing this result by 2 produces \( \theta = \pi/4 \). Because \( \pi/4 = 45^\circ \), you can conclude that the player must kick the football at an angle of \( 45^\circ \) so that the football will travel 200 feet.

c. From the model \( r = 200 \sin 2\theta \) you can see that the amplitude is 200. So the maximum range is \( r = 200 \) feet. From part (b), you know that this corresponds to an angle of \( 45^\circ \). Therefore, kicking the football at an angle of \( 45^\circ \) will produce a maximum horizontal distance of 200 feet.

**Checkpoint**

Now try Exercise 119.

**Writing About Mathematics**

**Deriving an Area Formula**

Describe how you can use a double-angle formula or a half-angle formula to derive a formula for the area of an isosceles triangle. Use a labeled sketch to illustrate your derivation. Then write two examples that show how your formula can be used.
5.5 Exercises

**VOCABULARY CHECK:** Fill in the blank to complete the trigonometric formula.

1. \( \sin 2u = \) ______
   2. \( \frac{1 + \cos 2u}{2} = \) ______

3. \( \cos 2u = \) ______
   4. \( \frac{1 - \cos 2u}{1 + \cos 2u} = \) ______

5. \( \sin \frac{u}{2} = \) ______
   6. \( \tan \frac{u}{2} = \) ______

7. \( \cos u \cos v = \) ______
   8. \( \sin u \cos v = \) ______

9. \( \sin u + \sin v = \) ______
   10. \( \cos u - \cos v = \) ______

**PREREQUISITE SKILLS REVIEW:** Practice and review algebra skills needed for this section at www.Eduspace.com.

In Exercises 1–8, use the figure to find the exact value of the trigonometric function.

\[ \sin \theta \]

1. \( \sin \theta \)
2. \( \tan \theta \)
3. \( \cos 2\theta \)
4. \( \sin 2\theta \)
5. \( \tan 2\theta \)
6. \( \sec 2\theta \)
7. \( \csc 2\theta \)
8. \( \cot 2\theta \)

In Exercises 9–18, find the exact solutions of the equation in the interval \([0, 2\pi]\).

9. \( 4 \sin x \cos x = 1 \)
10. \( \sin 2x + \cos x = 0 \)
11. \( 15 \tan 2x - \cot x = 0 \)
12. \( \sin 2x \sin x = \cos x \)
13. \( \cos 2x - \cos x = 0 \)
14. \( \cos 2x + \sin x = 0 \)
15. \( 12 \sin 2x - \cos x = 0 \)
16. \( \sin 2x + 2 \cos x = 0 \)
17. \( \sin 2x = -2 \sin 2x \)
18. \( \sin (2x + \cos 2x)^2 = 1 \)

In Exercises 19–22, use a double-angle formula to rewrite the expression.

19. \( 6 \sin x \cos x \)
20. \( 6 \cos^2 x - 3 \)
21. \( 4 - 8 \sin^2 x \)
22. \( (\cos x + \sin x)(\cos x - \sin x) \)

In Exercises 23–28, find the exact values of \( \sin 2u \), \( \cos 2u \), and \( \tan 2u \) using the double-angle formulas.

23. \( \sin u = -\frac{4}{5}, \pi < u < \frac{3\pi}{2} \)
24. \( \cos u = -\frac{2}{3}, \pi < u < \pi \)
25. \( \tan \frac{\theta}{2} \)
26. \( \sec \frac{\theta}{2} \)
27. \( \cot \frac{\theta}{2} \)
28. \( \sin \frac{\theta}{2} \)
29. \( \cos^4 x \)
30. \( \sin^8 x \)
31. \( \sin^2 x \cos^4 x \)
32. \( \sin^3 x \cos^4 x \)
33. \( \sin^2 x \cos^2 x \)
34. \( \sin^3 x \cos^2 x \)

In Exercises 29–34, use the power-reducing formulas to rewrite the expression in terms of the first power of the cosine.

35. \( \cos \frac{\theta}{2} \)
36. \( \sin \frac{\theta}{2} \)
37. \( \tan \frac{\theta}{2} \)
38. \( \sec \frac{\theta}{2} \)
39. \( \csc \frac{\theta}{2} \)
40. \( \cot \frac{\theta}{2} \)
In Exercises 41–48, use the half-angle formulas to determine the exact values of the sine, cosine, and tangent of the angle.

41. $75^\circ$  
42. $165^\circ$  
43. $112^\circ 30'$  
44. $67^\circ 30'$  
45. $\frac{\pi}{8}$  
46. $\frac{\pi}{12}$  
47. $\frac{3\pi}{8}$  
48. $\frac{7\pi}{12}$

In Exercises 49–54, find the exact values of $\sin(\theta/2)$, $\cos(\theta/2)$, and $\tan(\theta/2)$ using the half-angle formulas.

49. $\sin \theta = \frac{5}{13}$, $\pi/2 < \theta < \pi$  
50. $\cos \theta = \frac{3}{5}$, $0 < \theta < \frac{\pi}{2}$  
51. $\tan \theta = -\frac{5}{3}$, $\frac{3\pi}{2} < \theta < 2\pi$  
52. $\cot \theta = 3$, $\pi < \theta < \frac{3\pi}{2}$  
53. $\csc \theta = -\frac{5}{3}$, $\pi < \theta < \frac{3\pi}{2}$  
54. $\sec \theta = \frac{7}{2}$, $\frac{\pi}{2} < \theta < \pi$

In Exercises 55–58, use the half-angle formulas to simplify the expression.

55. $\sqrt{\frac{1 - \cos 6x}{2}}$  
56. $\sqrt{\frac{1 + \cos 4x}{2}}$  
57. $-\sqrt{\frac{1 - \cos 8x}{1 + \cos 8x}}$  
58. $-\sqrt{\frac{1 - \cos(\theta - 1)}{2}}$

In Exercises 59–62, find all solutions of the equation in the interval $[0, 2\pi)$. Use a graphing utility to graph the equation and verify the solutions.

59. $\sin \frac{x}{2} + \cos x = 0$  
60. $\sin \frac{x}{2} + \cos x - 1 = 0$  
61. $\cos \frac{x}{2} - \sin x = 0$  
62. $\tan \frac{x}{2} - \sin x = 0$

In Exercises 63–74, use the product-to-sum formulas to write the product as a sum or difference.

63. $6 \sin \frac{\pi}{4} \cos \frac{\pi}{4}$  
64. $4 \cos \frac{\pi}{3} \sin \frac{5\pi}{6}$  
65. $10 \cos 75^\circ \cos 15^\circ$  
66. $6 \sin 45^\circ \cos 15^\circ$  
67. $\cos 4\theta \sin 6\theta$  
68. $3 \sin 2\alpha \sin 3\alpha$  
69. $5 \cos (-5\beta) \cos 3\beta$  
70. $\cos 2\theta \cos 4\theta$

71. $\sin(x + y) \sin(x - y)$  
72. $\sin(x + y) \cos(x - y)$  
73. $\cos(\theta + \pi) \sin(\theta + \pi)$  
74. $\cos(\theta + \pi) \sin(\theta - \pi)$

In Exercises 75–82, use the sum-to-product formulas to write the sum or difference as a product.

75. $\sin 5\theta - \sin 3\theta$  
76. $\sin 3\theta + \sin \theta$  
77. $\cos 6x + \cos 2x$  
78. $\sin x + \sin 5x$  
79. $\sin(\alpha + \beta) - \sin(\alpha - \beta)$  
80. $\cos(\phi + 2\pi) + \cos \phi$

81. $\cos(\theta + \pi/2) - \cos(\theta - \pi/2)$  
82. $\sin(x + \pi/2) + \sin(x - \pi/2)$

In Exercises 83–86, use the sum-to-product formulas to find the exact value of the expression.

83. $\sin 60^\circ + \sin 30^\circ$  
84. $\cos 120^\circ + \cos 30^\circ$  
85. $\frac{3\pi}{4} - \cos \frac{\pi}{4}$  
86. $\sin \frac{5\pi}{4} - \sin \frac{3\pi}{4}$

In Exercises 87–90, find all solutions of the equation in the interval $[0, 2\pi]$. Use a graphing utility to graph the equation and verify the solutions.

87. $\sin 6x + \sin 2x = 0$  
88. $\cos 2x - \cos 6x = 0$  
89. $\frac{\cos 2x}{\sin 3x - \sin x} - 1 = 0$  
90. $\sin^2 3x - \sin^2 x = 0$

In Exercises 91–94, use the figure and trigonometric identities to find the exact value of the trigonometric function in two ways.

91. $\sin^2 \alpha$  
92. $\cos^2 \alpha$  
93. $\sin \alpha \cos \beta$  
94. $\cos \alpha \sin \beta$

In Exercises 95–110, verify the identity.

95. $\csc 2\theta = \frac{\csc \theta}{2 \cos \theta}$  
96. $\sec 2\theta = \frac{\sec^2 \theta}{2 - \sec^2 \theta}$  
97. $\cos^2 2\alpha - \sin^2 2\alpha = \cos 4\alpha$  
98. $\cos^4 x - \sin^4 x = \cos 2x$  
99. $(\sin x + \cos x)^2 = 1 + \sin 2x$
100. \( \sin \frac{\alpha}{3} \cos \frac{\alpha}{3} = \frac{1}{2} \sin \frac{2\alpha}{3} \)
101. \( 1 + \cos 10y = 2 \cos^2 5y \)
102. \( \frac{\cos 3\beta}{\cos \beta} = 1 - 4 \sin^2 \beta \)
103. \( \sec \frac{u}{2} = \pm \sqrt{\frac{2 \tan u}{\tan u + \sin u}} \)
104. \( \tan \frac{u}{2} = \csc u - \cot u \)
105. \( \frac{\sin x \pm \sin y}{\cos x + \cos y} = \tan \frac{x \pm y}{2} \)
106. \( \frac{\sin x + \sin y}{\cos x - \cos y} = -\cot \frac{x - y}{2} \)
107. \( \frac{\cos 4x + \cos 2x}{\sin 4x + \sin 2x} = \cot 3x \)
108. \( \frac{\cos t + \cos 3t}{\sin 3t - \sin t} = \cot t \)
109. \( \sin \left( \frac{\pi}{6} + x \right) + \sin \left( \frac{\pi}{6} - x \right) = \cos x \)
110. \( \cos \left( \frac{\pi}{3} + x \right) + \cos \left( \frac{\pi}{3} - x \right) = \cos x \)

In Exercises 111–114, use a graphing utility to verify the identity. Confirm that it is an identity algebraically.

111. \( \cos 3\beta = \cos^3 \beta - 3 \sin^2 \beta \cos \beta \)
112. \( \sin 4\beta = 4 \sin \beta \cos \beta (1 - 2 \sin^2 \beta) \)
113. \( (\cos 4x - \cos 2x)/(2 \sin 3x) = -\sin x \)
114. \( (\cos 3x - \cos x)/(\sin 3x - \sin x) = -\tan 2x \)

In Exercises 115 and 116, graph the function by hand in the interval \([0, 2\pi]\) by using the power-reducing formulas.

115. \( f(x) = \sin^2 x \)  
116. \( f(x) = \cos^2 x \)

In Exercises 117 and 118, write the trigonometric expression as an algebraic expression.

117. \( \sin(2 \arcsin x) \)  
118. \( \cos(2 \arccos x) \)

19. Projectile Motion The range of a projectile fired at an angle \( \theta \) with the horizontal and with an initial velocity of \( v_0 \) feet per second is

\[ r = \frac{1}{32} v_0^2 \sin 2\theta \]

where \( r \) is measured in feet. An athlete throws a javelin at 75 feet per second. At what angle must the athlete throw the javelin so that the javelin travels 130 feet?

120. Geometry The length of each of the two equal sides of an isosceles triangle is 10 meters (see figure). The angle between the two sides is \( \theta \).

![Isosceles Triangle]

(a) Write the area of the triangle as a function of \( \theta/2 \).
(b) Write the area of the triangle as a function of \( \theta \). Determine the value of \( \theta \) such that the area is a maximum.

Model It

121. Mach Number The mach number \( M \) of an airplane is the ratio of its speed to the speed of sound. When an airplane travels faster than the speed of sound, the sound waves form a cone behind the airplane (see figure). The mach number is related to the apex angle \( \theta \) of the cone by

\[ \sin \frac{\theta}{2} = \frac{1}{M} \]

(a) Find the angle \( \theta \) that corresponds to a mach number of 1.
(b) Find the angle \( \theta \) that corresponds to a mach number of 4.5.
(c) The speed of sound is about 760 miles per hour. Determine the speed of an object with the mach numbers from parts (a) and (b).
(d) Rewrite the equation in terms of \( \theta \).
122. **Railroad Track**  When two railroad tracks merge, the overlapping portions of the tracks are in the shapes of circular arcs (see figure). The radius of each arc \( r \) (in feet) and the angle \( \theta \) are related by

\[
x = 2r \sin \frac{\theta}{2}.
\]

Write a formula for \( x \) in terms of \( \cos \theta \).

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**Synthesis**

**True or False?** In Exercises 123 and 124, determine whether the statement is true or false. Justify your answer.

123. Because the sine function is an odd function, for a negative number \( u \), \( \sin 2u = -2 \sin u \cos u \).

124. \( \sin \frac{u}{2} = -\sqrt{\frac{1 - \cos u}{2}} \) when \( u \) is in the second quadrant.

In Exercises 125 and 126, (a) use a graphing utility to graph the function and approximate the maximum and minimum points on the graph in the interval \( [0, 2\pi] \) and (b) solve the trigonometric equation and verify that its solutions are the \( x \)-coordinates of the maximum and minimum points of \( f \). (Calculus is required to find the trigonometric equation.)

| Function \( f(x) = 4 \sin \frac{x}{2} + \cos x \) | Trigonometric Equation \( 2 \cos \frac{x}{2} - \sin x = 0 \) |
| Function \( f(x) = \cos 2x - 2 \sin x \) | Trigonometric Equation \( -2 \cos x(2 \sin x + 1) = 0 \) |

**Exploration** Consider the function given by

\[
f(x) = \sin^4 x + \cos^4 x.
\]

(a) Use the power-reducing formulas to write the function in terms of cosine to the first power.

(b) Determine another way of rewriting the function. Use a graphing utility to rule out incorrectly rewritten functions.

(c) Add a trigonometric term to the function so that it becomes a perfect square trinomial. Rewrite the function as a perfect square trinomial minus the term that you added. Use a graphing utility to rule out incorrectly rewritten functions.

(d) Rewrite the result of part (c) in terms of the sine of a double angle. Use a graphing utility to rule out incorrectly rewritten functions.

(e) When you rewrite a trigonometric expression, the result may not be the same as a friend’s. Does this mean that one of you is wrong? Explain.

**Conjecture** Consider the function given by

\[
f(x) = 2 \sin \left(2 \cos^2 \frac{x}{2} - 1\right).
\]

(a) Use a graphing utility to graph the function.

(b) Make a conjecture about the function that is an identity with \( f \).

(c) Verify your conjecture analytically.

**Skills Review**

In Exercises 129–132, (a) plot the points, (b) find the distance between the points, and (c) find the midpoint of the line segment connecting the points.

| (a) \((5, 2), (-1, 4)\) | (b) \((5, -3), (6, 10)\) |

In Exercises 133–136, find (if possible) the complement and supplement of each angle.

| (a) \(55^\circ\) | (b) \(162^\circ\) |
| (a) \(109^\circ\) | (b) \(78^\circ\) |

135. (a) \(\frac{\pi}{18}\)  (b) \(\frac{9\pi}{20}\)

136. (a) 0.95  (b) 2.76

137. **Profit** The total profit for a car manufacturer in October was 16% higher than it was in September. The total profit for the 2 months was $507,600. Find the profit for each month.

138. **Mixture Problem** A 55-gallon barrel contains a mixture with a concentration of 30%. How much of this mixture must be withdrawn and replaced by 100% concentrate to bring the mixture up to 50% concentration?

139. **Distance** A baseball diamond has the shape of a square in which the distance between each of the consecutive bases is 90 feet. Approximate the straight-line distance from home plate to second base.